Homework 4
Math 117 - Summer 2022

1) Consider $V = \mathbb{R}^3$ with standard basis e_1, e_2, e_3. Recall from class we have an isomorphism

$$V \otimes V^* \to \mathcal{L}(V, V) \simeq M_{3\times 3}(\mathbb{R})$$

and that we denote the linear map we get under this isomorphism just by

$$v \otimes \varphi$$

for $v \in V, \varphi \in V^*$. Let $\varphi(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = x + z$

(a) (2 points) Find the linear map (ie matrix) associated to the tensor $e_2 \otimes \varphi$

(b) (2 points) Find the linear map (ie matrix) associated to the tensor $e_1 \otimes \varphi$

(c) (1 point) Find the linear map (ie matrix) associated to the tensor $(2e_1 + 3e_2) \otimes \varphi$

Solution:

2) (3 points) Recall that for any real vector space V we defined the "conjugation" map on the complexification

$$\tau : \mathbb{C} \otimes V \to \mathbb{C} \otimes V$$

Prove that the following diagram commutes

$$\begin{array}{ccc} \mathbb{C} \otimes \mathbb{R}^n & \xrightarrow{\sim} & \mathbb{C}^n \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \overline{(-)}} \\ \mathbb{C} \otimes \mathbb{R}^n & \xrightarrow[\sim]{} & \mathbb{C}^n \end{array}$$

where $\overline{(-)} : \mathbb{C}^n \to \mathbb{C}^n$ is the more usual conjugation map, and where $\mathbb{C} \otimes \mathbb{R}^n \xrightarrow{\sim} \mathbb{C}^n$ is the isomorphism we constructed in class

Solution:

3) Let V be a vector space and denote $V^k := V \times \cdots \times V$ as the cartesian product of itself k times. Now let

$$f : V^k \to W$$

be an alternating multilinear map.

(a) (2 points) Prove that if $v_i \in span(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k)$ then

$$f(v_1, v_2, \ldots, v_i, \ldots, v_k) = 0$$

(b) (2 points) Suppose now that dim(V)=n. Prove that $\bigwedge^m(V) = 0$ for $m > n$

Solution:

4) Let $V = M_{2\times 2}(\mathbb{C})$ with standard basis $\mathcal{B} = (m_1, m_2, m_3, m_4)$ and let $T : V \to V$ be the map

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a-d & d \\ b+c & c \end{pmatrix}$$

(a) (1 point) Write down a basis for $\bigwedge^2(V)$, $\bigwedge^3(V)$ and $\bigwedge^4(V)$

(b) (2 points) Find the matrix for $\bigwedge^3(T) : \bigwedge^3(V) \to \bigwedge^3(V)$ (hint: it will be a 4×4 matrix)

(c) (1 point) Find the determinant of T

Solution:

5) (4 points) Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

be a 3×3 real matrix and consider it as a linear map

$$A : \mathbb{R}^3 \to \mathbb{R}^3$$

Compute $\bigwedge^3(A) : \bigwedge^3(\mathbb{R}^3) \to \bigwedge^3(\mathbb{R}^3)$ with respect to the standard basis of $\mathbb{R}^3$ and verify that the usual formula for the determinant of a 3×3 matrix holds

Solution: